

Bangalore August 2010.

UNITARY DOUBLE PRODUCT INTEGRALS AS BOGOLUBOV IMPLEMENTORS (The pseudo-rotation case)

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Joint work with Paul Jones.

1. Double product integrals in QSC
2. Bogolubov transformations.
3. A programme to construct unitary double product integrals as Bogolubov implementors (the pseudo-rotation case).
4. Step 1 - Discrete approximation
5. Step 2 - Simple Bogolubov transformation
6. Step 3 - A discrete double product and two limits
7. The miracle.
8. Three rigorous theorems.

1. Double product integrals.

Notation $\mathcal{G}$ is the Itô algebra

$$\mathcal{G} = \mathbb{C}\langle dA^t, dA, dT \rangle$$

	dA^t	dA	dT
dA^t	0	0	0
dA	dT	0	0
dT	0	0	0

$$dr \in \mathcal{G} \otimes \mathcal{G}.$$

$$a < b, s < t \in \mathbb{R}_+$$

Definition The (forward) rectangular double product integral over $]a, b] \times]s, t]$

$${}_a^b \Pi_s^t (1 + dr) =$$

$$\textcircled{1} {}_a^b \Pi \left(1 + \hat{\Pi}_s^t (1 + dr) \right)$$

$$\textcircled{2} \hat{\Pi}_s^t \left(1 + {}_a^b \hat{\Pi} (1 + dr) \right).$$

Here :

$$\hat{\Pi}_s^t(1+dr)$$

$$= \sum_{n=1}^{\infty} \int_{s < x_1 < \dots < x_n < t} dr(\cdot, x_1) \dots dr(\cdot, x_n)$$

$$\in \mathcal{G} \otimes \mathcal{P}_s^t$$

where $\mathcal{P}_s^t$ is the unital algebra of iterated integrals over $]s, t]$.

Thus we can define

$${}_a^b \Pi \left(1 + \hat{\Pi}_s^t(1+dr) \right) \text{ as the solution}$$

at b of the stochastic differential equation (with right system algebra $\mathcal{P}_s^t$)

$$dX = X \hat{\Pi}_s^t(1+dr), \quad X(s) = 1.$$

Similarly:

$${}^b\hat{\Pi}_a(1+dr)$$

$$= \sum_{n=1}^{\infty} \left\{ dr(x_1, \cdot) dr(x_2, \cdot) \dots dr(x_n, \cdot) \right. \\ \left. a < x_1 < \dots < x_n < b \right.$$

$\in \mathcal{P}_a^b \otimes \mathcal{G}$ and we can define

$\Pi_s^t(1 + {}^b\hat{\Pi}_a(1+dr))$ as the solution

at t of the sde (with left system algebra $\mathcal{P}_a^b$)

$$dX = X {}^b\hat{\Pi}_a(1+dr), \quad X(s) = 1.$$

"Fubini theorem" The two

definitions agree:

$${}^b\hat{\Pi}_a(1 + \hat{\Pi}_s^t(1+dr)) = \Pi_s^t(1 + {}^b\hat{\Pi}_a(1+dr))$$

Proof: Using multiplication rule in $\mathcal{P}_a^b$

$$\textcircled{1} = 1 + \sum_{n,n=1}^{\infty} I_a^b \otimes I_s^t \left(\sum \prod_{k=1}^n dr^{A_k, m+k} \right) \left(A_1 \cup \dots \cup A_n = N_n \right)$$

$$\textcircled{2} = 1 + \sum_{n,n=1}^{\infty} I_a^b \otimes I_s^t \left(\sum \prod_{j=1}^n dr^{j, m+\theta_j} \right) \left(B_1 \cup \dots \cup B_n = N_n \right)$$

both of which are equal to

$$\textcircled{3} 1 + \sum_{n,n=1}^{\infty} I_a^b \otimes I_s^t \left(\sum \prod_{(j,k) \in N_n \times N_n} (dr)^{M_{j,k}} \right)^{n,n}$$

where

$$M_{j,k} = \begin{cases} 1 \\ 0 \end{cases} \Leftrightarrow \begin{cases} j \in A_k \\ j \notin A_k \end{cases} \Leftrightarrow \begin{cases} k \in B_j \\ k \notin B_j \end{cases}$$

Three Properties

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1. Bievolution

${}_a^b \Pi_s^t (1+dr)$ is an evolution in (a,b) for fixed (s,t) and vice versa

2. Bicovariance under shifts

$${}_{a+c}^{b+c} \Pi_{s+u}^{t+u} (1+dr)$$

$$= (\Gamma_c \otimes \Gamma_u)^* {}_a^b \Pi_s^t (1+dr) (\Gamma_c \otimes \Gamma_u)$$

where Γ_c is the second-quantised forward shift.

3. Bicovariance under time reversal

$${}_a^b \Pi_s^t (1+dr)$$

$$= (R_a^b \otimes R_s^t)^* {}_a^b \Pi_s^t (1+dr) (R_a^b \otimes R_s^t)$$

$$R_a^b e(f) = e(\bar{f}_a^b), \bar{f}_a^b(x) = \begin{cases} f(a+b-x) & x \in]a,b[\\ f(x) & x \notin]a,b[\end{cases}$$

Bogolubov transformations 7.

Let h be a Hilbert space and let $\mathcal{F}(h)$ be the Fock space over h . We have the Weyl RCCR by unitary operators $W(h)$, $h \in h$ satisfying

$$W(f)W(g) = e^{-i \operatorname{Im} \langle f, g \rangle} W(f+g).$$

Definition A Bogolubov transformation is an invertible real-linear map $B: h \rightarrow h$ which preserves $\operatorname{Im} \langle \cdot, \cdot \rangle$.

Thus $W_B(\cdot) =: W(B^{-1} \cdot)$ is a new RCCR. If there is a unitary U_B on $\mathcal{F}(h)$ such that

$$W_B(\cdot) = U_B^{-1} W(\cdot) U_B$$

then B is unitarily implemented by U_B .

Example If B is unitary it is Bogolubov and it is unitarily implemented by $\Gamma(B)$

Programme Exhibit all unitary double products as implementors of explicit Bogolubov transformations.

This programme has been completed for the "rotational case",

$$^b \Pi^t, (1 + \lambda (dA^+ \otimes dA - dA \otimes dA^+)).$$

The Bogolubov transformation is a unitary, related to the old "time-orthogonal unitary dilation"

RLH, PDF Ion + KR Parthasarathy
CMP 83, 261-280 (1982). See

1) RLH: A double dilation constructed from a double product of rotations

Markov Processes + RF 13 (2007)

2) RLH: An analytic double product integral $QP = I \, dA \, \overline{XX}$ (2007)

Now I want to consider the 9.
pseudorotational case where

$$dr = \lambda (dA^\dagger \otimes dA^\dagger - dA \otimes dA) + \frac{\lambda^2}{2} dT \otimes dT$$

$$\Rightarrow dr + dr^\dagger + dr dr^\dagger = 0$$

$$\Rightarrow {}^b_a \Pi_s^t (1 + dr) \text{ is unitary.}$$

Step 1 Equipartition the interval

$$a = x_0 < x_1 < \dots < x_m = b, \quad x_j = a + \frac{j}{m}(b-a)$$

$$s = y_0 < y_1 < \dots < y_n = t, \quad y_k = s + \frac{k}{n}(t-s)$$

Approximate ${}^b_a \Pi_s^t (1 + dr)$ by

$$\prod_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n} \left(1 + \lambda \sqrt{\frac{(b-a)(t-s)}{mn}} (a_j^\dagger \otimes b_k^\dagger - a_j \otimes b_k) + \frac{\lambda^2}{2mn} (b-a)(t-s) \right)$$

where $(a_j^\dagger, a_j), (b_k^\dagger, b_k)$ are the standard canonical pairs

$$a_j^\# = \sqrt{\frac{m}{b-a}} (A^\#(x_j) - A^\#(x_{j-1}))$$

$$b_k^\# = \sqrt{\frac{n}{t-s}} (A^\#(y_k) - A^\#(y_{k-1}))$$

Step 2 Consider the Bogolubov¹⁰. transformation for two degrees of freedom implemented by the (approximately) unitary operator forming one term in the discrete double product

$$(1 + \phi_{m,n} (a^\dagger b^\dagger - ab)) \\ \approx \exp \phi_{m,n} (a^\dagger b^\dagger - ab)$$

where $\phi_{m,n} = \lambda \sqrt{\frac{(b-u)(t-s)}{mn}}$

and we ignore the constant time term.

We have, since $[a, a^\dagger] = [b, b^\dagger] = 1$,

$$\text{ad}_{a^\dagger b^\dagger - ab} (a) = -b^\dagger$$

$$\text{ad}_{a^\dagger b^\dagger - ab} (b^\dagger) = a$$

$$\therefore \text{Ad}_{\exp \phi_{m,n} (a^\dagger b^\dagger - ab)} (a) \\ = \exp \text{ad}_{\phi_{m,n} (a^\dagger b^\dagger - ab)} (a)$$

$$= \cosh \phi_{m,n} a - \sinh \phi_{m,n} b^\dagger$$

and similarly

$$\text{Ad}_{\exp \phi_{m,n}(a^\dagger b^\dagger - ab)}(b)$$

$$= \sinh \phi_{m,n} a^\dagger - \cosh \phi_{m,n} b$$

Thus the Bogolubov transformation is the invertible real-linear transformation in $\mathbb{C}^2$

$$\theta_{m,n} \begin{pmatrix} z \\ w \end{pmatrix} \mapsto \begin{pmatrix} \alpha & \beta \kappa \\ \gamma \kappa & \delta \end{pmatrix} \begin{pmatrix} z \\ w \end{pmatrix}$$

where $\alpha = \delta = \cosh \phi_{m,n}$

$$\beta = \gamma = -\sinh \phi_{m,n}$$

and κ is complex conjugation.

Step 3 Consider the product of Bogolubov transformations in $\mathbb{C}^{m+n}$

$$\theta^{(m,n)} = \prod_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n} \theta_{m,n}^{(j,k)} \quad \text{where:}$$

$$\beta_{m,n}^{(j,k)} = \begin{bmatrix} & (j) & & (m+k) & & n \\ & \vdots & & \vdots & & \\ (j) & \dots & \alpha & \dots & \beta_k & \dots \\ & \vdots & & \vdots & & \\ (m+k) & \dots & \gamma_k & \dots & \delta & \dots \\ & \vdots & & \vdots & & \end{bmatrix}$$

is the $(m+n) \times (m+n)$ matrix
 got by embedding $B = \begin{pmatrix} \alpha & \beta_k \\ \gamma_k & \delta \end{pmatrix}$
 in the intersections of the
 j^{th} and $(m+k)^{\text{th}}$ rows and columns
 and filling in 1's on the diagonal
 and 0's elsewhere.

I do not have an explicit
 formula for $B^{(m,n)} = \prod_{(j,k) \in \mathbb{N}_m \times \mathbb{N}_n} \beta_{m,n}^{(j,k)}$.
 But (thank you Paul) I do
 know that:

$$B^{(1,n)} = \begin{bmatrix} \alpha^n & \beta \kappa & \alpha \beta \kappa & \alpha^2 \beta \kappa & \dots & \alpha^{n-1} \beta \kappa & \alpha^{n-1} \beta \kappa \\ \delta \alpha^{n-1} \kappa & \delta & \delta \beta & \delta \alpha \beta & \dots & \delta \alpha^{n-2} \beta & \delta \alpha^{n-2} \beta \\ \delta \alpha^{n-2} \kappa & 0 & \delta & \delta \beta & \dots & \delta \alpha^{n-3} \beta & \delta \alpha^{n-3} \beta \\ \delta \alpha^{n-3} \kappa & 0 & 0 & \delta & \dots & \delta \alpha^{n-4} \beta & \delta \alpha^{n-4} \beta \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta \alpha \kappa & 0 & 0 & \dots & 0 & \delta & \delta \beta \\ \delta \kappa & 0 & 0 & \dots & 0 & 0 & \delta \end{bmatrix} \quad |3$$

$$B^{(m,1)} = \begin{bmatrix} \alpha & 0 & 0 & \dots & 0 & 0 & \beta \kappa \\ \beta \delta & \alpha & 0 & \dots & 0 & 0 & \beta \delta \kappa \\ \beta \delta \delta & \beta \delta & \alpha & \dots & 0 & 0 & \beta \delta^2 \kappa \\ \beta \delta^2 \delta & \beta \delta^2 & \beta \delta & \dots & 0 & 0 & \beta \delta^3 \kappa \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ \beta \delta^{n-2} \delta & \beta \delta^{n-2} & \beta \delta^{n-3} & \dots & \beta \delta & \alpha & \beta \delta^{n-1} \kappa \\ \delta^{n-2} \kappa & \delta^{n-2} \kappa & \delta^{n-3} \kappa & \dots & \delta \kappa & \delta \kappa & \delta^m \end{bmatrix}$$

Embedding C^n in $L^2(\mathbb{R})$ by $\epsilon_j \mapsto \sqrt{\frac{n}{s-t}} \chi_{x_j}^{x_j}$

$B^{(1,n)}$ becomes an operator in $C \oplus L^2(\mathbb{R})$

and we can form

$$\lim_{n \rightarrow \infty} B^{(1,n)} = \begin{bmatrix} \alpha_s^t & \beta_s^t \\ \delta_s^t \kappa & \delta_s^t \end{bmatrix} \quad \text{and similarly}$$

$$\lim_{m \rightarrow \infty} B^{(m,1)} = \begin{bmatrix} \alpha & \beta \\ \delta \kappa & \delta \end{bmatrix} \quad \text{where, for example,}$$

$$\alpha_s^t = \lim_n \alpha^n = \lim_n \left(\cosh \left(\lambda \sqrt{\frac{(b-a)(t-s)}{mn}} \right) \right) \\ = e^{\frac{\lambda^2}{2m} (b-a)(t-s)}$$

Similarly β_s^t is an explicit bra $\langle f_s^t |$, δ_s^t is an explicit ket $|g_s^t\rangle$ and δ_s^t an explicit operator on $L^2(\mathbb{R})$, namely

$$f_s^t(x) = X_s^t(x) e^{\frac{\lambda^t}{2m}(b-a)(x-s)} \quad 16$$

$$g_s^t(x) = X_s^t(x) e^{\frac{\lambda^t}{2m}(b-a)(t-x)}$$

$\delta_s^t = 1 + \text{integral operator with}$
 kernel $(x,y) \mapsto \langle_s^t(x,y) e^{\frac{\lambda^t}{2m}(b-a)(y-x)}$.

Similar calculations in the rotational case led to the HIP TOVD (and on to quantum stochastic calculus).

THE MIRACLE

$\begin{bmatrix} \alpha & B\kappa \\ \delta\kappa & \delta \end{bmatrix}$ can be replaced by

$\begin{bmatrix} \alpha_s^t & B_s^t \kappa \\ \delta_s^t \kappa & \delta_s^t \end{bmatrix}$ in forming $\rho^{(n,1)}$

and the limit $\lim_{n \rightarrow \infty} \rho^{(n,1)}$

$= \lim_{n \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \prod_{j,k} \rho_{n,n}^{(j,k)} \right)$ found $\leftarrow$

The result is that:

$$\lim_{n \rightarrow \infty} \lim_{\lambda \rightarrow 0} \Pi_{(j,k)}^{(j)} \begin{pmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ 0 & \dots & \cosh \phi_{jk} & \dots & \sinh \phi_{jk} & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & \sinh \phi_{jk} & \dots & \cosh \phi_{jk} & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

(with $\phi_{jk} = \lambda \int_a^b V_a^b(x) dx$)

$$= \begin{bmatrix} e^{\lambda(s-t)} V_a^b & -\lambda F \kappa \\ -\lambda G \kappa & e^{\lambda(b-a)} V_s^t \end{bmatrix}$$

where V_a^b is the Volterra operator on the interval $[a, b]$

$$V_a^b F(x) = \chi_a^b(x) \int_a^b F(y) dy$$

$$(F F)(x) = \lambda \chi_a^b(x) \int_a^b \langle \chi_s^t, e^{\lambda \int_s^t V_s^t} \chi_s^t \rangle \chi_s^t dy$$

$$(G F)(x) = \lambda \chi_s^t(x) \int_a^b \langle \chi_a^b, e^{\lambda \int_a^b V_a^b} \chi_a^b \rangle \chi_a^b dy.$$

We now have to prove

16.

Theorem 1 The real-linear operator in $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$

$${}^b\mathcal{B}_a^t = \begin{bmatrix} e^{\lambda^*(s-t)} V_a^b & -\lambda F K \\ -\lambda G K & e^{\lambda^*(b-a)} V_s^t \end{bmatrix}$$

is Bogolubov. Moreover it is an evolution in (a, b) for fixed (s, t) and vice-versa, and it is bicovariant under shifts and time reversals.

Theorem 2 Each ${}^b\mathcal{B}_a^t$ is unitarily implementable. Moreover there is a unique choice ${}^b\mathcal{U}_a^t$ of each implementor such that $({}^b\mathcal{U}_a^t)$ is a bievolution bicovariant under shifts and time reversals

[For the last part we use the method of HIF II, in PRIMS Kyoto 1984]

Theorem 3 For this choice,

$${}^b\mathcal{U}_a^t = {}^b\prod_a^t \left(1 + \lambda \left(dA^+ dA^+ - dA dA \right) + \frac{\lambda^2}{2} dT dT \right)$$